Strategies for inversion of the additive relationship matrix among genotyped animals

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Introduction: The case of $A_{22}$ vs. $A$

$A_{22} = \text{subpart of } A \text{ whose inversion is required, e.g. in ssGBLUP}$
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✓ The inverse of $A$ is computed as a sum of vector products (Henderson, 1976)

$$A_{(i)}^{-1} = \begin{bmatrix} A_{(i-1)}^{-1} & 0 \\ 0' & 0 \end{bmatrix} + \alpha_{(i)} \begin{bmatrix} -b_{(i)} \\ 1 \end{bmatrix} \begin{bmatrix} -b'_{(i)} \\ 1 \end{bmatrix}$$
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✓ The inverse of $A$ is computed as a sum of vector products (Henderson, 1976)

$$A^{-1}_{(i)} = A^{-1}_{(i-1)} 0 \quad 0' + \alpha_{(i)} \begin{bmatrix} -b_{(i)} \\ 0' \end{bmatrix}$$

$$A^{-1}_{(i)} = \left( T^{-1}_{A(i)} \right)' D^{-1}_{A(i)} T^{-1}_{A(i)}$$

$$T^{-1}_{A(i)} = \begin{bmatrix} T^{-1}_{A(i-1)} & 0 \\ -b'_{(i)} & 1 \end{bmatrix}$$

$$D^{-1}_{A(i)} = \begin{bmatrix} D^{-1}_{A(i-1)} & 0 \\ 0' & \alpha_{(i)} \end{bmatrix}$$
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✓ $A_{22}$ = subpart of $A$ whose inversion is required, e.g. in ssGBLUP

![Diagram showing the relationship between $A$, $A_{22}$, genotyped, and non-genotyped individuals]
Sparsity in the inverse factor of $A_{22}$

Example: An animal and its parents
Sparsity in the inverse factor of $A_{22}$

- Example: An animal and its parents

The diagram illustrates the inverse matrix $T^{-1}$ with elements $A$, $B$, and $C$, showing non-zero elements for $\neq 0$, zero elements for $0$, and unspecified elements for $\?$. The connections indicate relationships between the elements.
Issues and Objective

✓ How sparse is the inverse of $A_{22}$?

... How sparse is the inverse factor ($T^{-1}$) of $A_{22}$?

✓ How a putative sparsity could be used in computation of the inverse?

→ Main objective: To avoid useless computations
Sparsity in the inverse factor of $A_{22}$

✓ How to deal with more complex cases?

✓ By a comprehensive search in the pedigree
  ✓ « SP Algorithm »
  ✓ Explores pedigree branches and apply simple rules
  ✓ Uses only pedigree and incidence vector
  ✓ Returns a symbolic inverse factorization
Sparsity in the inverse factor of $A_{22}$

✓ Some performances on different sizes of $A_{22}$:

![Graph showing performance times for different sizes of $A_{22}$ with two curves for $g = 5$ and $g = 10$.]

Processor: Intel Xeon 64-bit
64Gb of RAM,
Clock speed: 3.16GHz
Strategies to take sparsity into account

1. Successive construction of the inverse

\[
\mathbf{A}^{-1}_{22(i)} = \begin{bmatrix}
\mathbf{A}^{-1}_{22(i-1)} & 0 \\
0' & 0
\end{bmatrix} + \alpha(i) \begin{bmatrix}
-b_{(i)} \\
1
\end{bmatrix} \begin{bmatrix}
-b'_{(i)} \\
1
\end{bmatrix}
\]

How to get \( b \)?

1. \( b_{(i)} = \mathbf{A}^{-1}_{22(i-1)} \mathbf{A}_{22(i-1)} (\cdot, 1 : i-1) \)
2. \( \mathbf{A}_{22(i-1)} b_{(i)} = \mathbf{A}_{22(i-1)} (\cdot, 1 : i-1) \)
Strategies to take sparsity into account

1. Restricting the product only to elements of $b$ different from 0

$$b_{(i)} = A^{-1}_{22(i-1)}A_{22(i-1)}(:, 1:i-1) \rightarrow x = Z y$$
Strategies to take sparsity into account

2. Solving a linear system of lower size

\[ A_{22(i-1)} b_{(i)} = A_{22(i-1)}(:,1:i-1) \rightarrow Zx = y \]
Strategies to take sparsity into account

Processor: Intel Xeon 64-bit
8Gb of RAM,
Clock speed: 3 GHz
Strategies to take sparsity into account

✓ Order of $A_{22} = \text{Number of genotyped animals}$
✓ Depends on the pedigree (depth, lines, ...)

![Graph showing the percentage of zeros in relation to the number of genotyped animals]
Strategies to take sparsity into account

3. Storing the inverse of $A_{22}$ from time to time and updating this inverse only for recent animals

$$A_{22(t+1)}^{-1} = \begin{bmatrix} A_{22(t)}^{-1} & 0 \\ 0' & 0 \end{bmatrix} + \alpha(x) \begin{bmatrix} -b(x) \\ 1 \end{bmatrix} \begin{bmatrix} -b'(x) & 1 \end{bmatrix}$$
Strategies to take sparsity into account

A

$A_{22(t)}$

$A_{22(t+1)}$
Strategies to take sparsity into account

A

A_{22(t)}

A_{22(t+1)}
Take-home messages

1. Sparsity pattern of the inverse of $A_{22}$ can be set up without matrix computations, even for large matrices
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2. Using sparsity reduces time for inversion, if that inversion uses the inverse factor.
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1. Sparsity pattern of the inverse of $A_{22}$ can be set up without matrix computations, even for large matrices

2. Using sparsity reduces time for inversion, if that inversion uses the inverse factor

3. As the order of $A_{22}$ increases, inversion shrinks to solve multiple small linear systems that are identified by SP algorithm
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